PSTAT 5A Practice Worksheet 5 - SOLUTIONS

Continuous Random Variables and Confidence Intervals

Instructor Solutions

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# 1. Section A Solutions: Continuous Random Variables

*Solution*. **Solution A1: Distribution Identification and Properties**

**(a) Exponential Distribution**

Since the average time between arrivals is $2$ minutes, we have:

* **Parameter** $λ$**:** The rate parameter $λ=\frac{1}{μ}=\frac{1}{2}=0.5$ arrivals per minute
* **Probability calculation:** $P\left(X\leq 1\right)$ where $X∼Exponential\left(0.5\right)$
* For exponential distribution: $P\left(X\leq x\right)=1−e^{\left(−λx\right)}$
* $P\left(X\leq 1\right)=1−e^{\left(−0.5×1\right)}=1−e^{\left(−0.5\right)}=1−0.6065=$

**(b) Uniform Distribution**

* **Parameters:** $a=10,b=30$
* **Expected Value:** $E\left[X\right]=\left(a+b\right)/2=\frac{\left(10+30\right)}{2}=$
* **Variance:** $Var\left(X\right)=\frac{\left(b−a\right)^{2}}{12}=\frac{\left(30−10\right)^{2}}{12}=\frac{400}{12}=$

*Solution*. **Solution A2: Normal Distribution Calculations**

Given: $X∼N\left(64,2.5^{2}\right)$

**(a)** $P\left(X>67\right)$

**Step 1: Standardize**

$Z=\left(67−64\right)/2.5=3/2.5=1.2$

**Step 2: Find probability**

$P\left(X>67\right)=P\left(Z>1.2\right)=1−P\left(Z\leq 1.2\right)=1−0.8849=$

**(b) 25th percentile**

Step 1: Find $z$-value for $25$ -th percentile

$P\left(Z\leq z\right)=0.25$, so $z\_{0.25}=−0.6745$

Step 2: Convert back to $X$

$x=μ+zσ=64+\left(−0.6745\right)\left(2.5\right)=64−1.6863= inches$

**(c) P(62 < X < 68)**

Step 1: Standardize both values

$Z\_{1}=\left(62−64\right)/2.5=−0.8$ $Z\_{2}=\left(68−64\right)/2.5=1.6$

Step 2: Find probability

$P\left(62<X<68\right)=P\left(−0.8<Z<1.6\right)=P\left(Z<1.6\right)−P\left(Z<−0.8\right)$

$=0.9452−0.2119=$

# 2. Section B Solutions: Confidence Intervals

*Solution*. **Solution B1: Understanding Confidence Intervals**

**(a) Explanation of** $95\%$ **Confidence Interval:**

A $95\%$ confidence interval means that if we were to repeat our sampling process many times (say $100$ times) and construct a confidence interval each time using the same method, approximately $95$ of those intervals would contain the true population mean. **It does NOT mean there’s a** $95\%$ **probability that the population mean lies in any one specific interval.**

**(b) Sample mean and margin of error:**

Given $CI$: ($150g,170g$)

* **Sample mean:** $‾=\left(150+170\right)/2=$
* **Margin of error:** $E=\left(170−150\right)/2=$

**(c) True or False statement:**

**FALSE.** Once we calculate a specific confidence interval, the population mean either is or isn’t in that interval, there’s no probability involved for that specific interval. The $95\%$ refers to the long-run success rate of the method, not the probability for any individual interval.

*Solution*. **Solution B2: Constructing Confidence Intervals**

Given: $n=36,‾=78.5,s=12$

**(a) 95% Confidence Interval:**

Step 1: Check conditions

* $n=36\geq 30$, so we can use $z$-distribution
* For $95$

Step 2: Calculate margin of error

$E=z\_{0.025}×\left(\frac{s}{\sqrt{n}}\right)=1.96×\left(\frac{12}{\sqrt{36}}\right)=1.96×\left(\frac{12}{6}\right)=1.96×2=3.92$

Step 3: Construct interval

$CI=‾\pm E=78.5\pm 3.92=$

**(b) Interpretation:**

We are $95\%$ confident that the true population mean test score is between $74.58$ and $82.42$ points.

**(c) Effects on interval width:**

* **Increasing confidence level to 99%:** The interval would become **wider** because we need $z\_{0.005}=2.576>1.96$
* **Increasing sample size to 144:** The interval would become **narrower** because the margin of error would be $E=1.96×\left(\frac{12}{\sqrt{144}}\right)=1.96×1=1.96$ (smaller than $3.92$)

*Solution*. **Solution B3: Sample Size Determination**

Given: $E=\$5$, confidence = $95\%,σ=\$25$

**(a) Required sample size:**

Step 1: Use sample size formula

$n=\left(z\_{0.025}×\frac{σ}{E}\right)^{2}$

Step 2: Substitute values

$n=\left(1.96×25/5\right)^{2}=\left(49/5\right)^{2}=9.8^{2}=96.04$

Step 3: Round up

$$ customers (**always round up for sample size**)

**(b) For margin of error = $3:**

$n=\left(1.96×25/3\right)^{2}=\left(49/3\right)^{2}=16.333^{2}=266.67$

$$ customers

# 3. Optional Problem Solutions

*Solution*. **Optional Solution 1: Conceptual Understanding**

**(a) Differences between discrete and continuous:**

**Values they can take:**

* Discrete: Countable values (integers, specific points)
* Continuous: Uncountably infinite values (any real number in an interval)

**How we calculate probabilities:**

* Discrete: $P\left(X=x\right)$ can be non-zero; we sum probabilities
* Continuous: $P\left(X=x\right)=0$ for any specific $x$; we integrate over intervals

**(b) Why** $P\left(X=x\right)=0$ **for continuous distributions:**

In continuous distributions, there are infinitely many possible values in any interval. The probability of hitting any one exact value is infinitesimally small, hence zero. We instead calculate $P\left(a<X<b\right)$ by integrating the PDF over the interval $\left[a,b\right]$.

**(c) Relationship between PDF and CDF:**

* **PDF (f(x)):** The probability density function gives the “density” of probability at each point
* **CDF (F(x)):** The cumulative distribution function gives $P\left(X\leq x\right)$
* **Relationship:** $F\left(x\right)=\int\_{−\infty }^{x}f\left(t\right)dt, and f\left(x\right)=F′\left(x\right)$

**Key Takeaways:**

1. **Always standardize** normal distribution problems using $Z=\frac{\left(X−μ\right)}{σ}$
2. **Interpret confidence intervals** in context, they’re about the method’s reliability, not individual interval probabilities
3. **Choose the right distribution** use $t$ when $σ$ is unknown and $n<30$
4. **Round up sample sizes** to ensure you meet the margin of error requirement
5. **For continuous distributions**, focus on intervals, not individual points