# 1. Prize Wheel

**Given:** A prize wheel with the following distribution:

| Prize ($x$) | \$0 | \$5 | \$10 | \$25 |
|---|---|---|---|---|
| Probability $P(x)$ | 0.40 | 0.35 | 0.20 | 0.05 |

**Solution:**

$$\begin{aligned} E[X] &= \sum x \cdot P(x) \\ &= 0(0.40) + 5(0.35) + 10(0.20) + 25(0.05) \\ &= 0 + 1.75 + 2.00 + 1.25 \\ &= \boxed{\$5.00} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \sum x^2 \cdot P(x) \\ &= 0^2(0.40) + 5^2(0.35) + 10^2(0.20) + 25^2(0.05) \\ &= 0 + 25(0.35) + 100(0.20) + 625(0.05) \\ &= 0 + 8.75 + 20.00 + 31.25 \\ &= 60.00 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - [E[X]]^2 \\ &= 60.00 - (5.00)^2 \\ &= 60.00 - 25.00 \\ &= \boxed{35.00} \end{aligned}$$

On average, a typical spin yields \$5.00. The variance of 35.00 indicates considerable variability in the prize amounts.

# 2. SAT vs ACT Performance Comparison

**Given:**

- Ann's SAT: 1300 (SAT scores: $\mu = 1100$, $\sigma = 200$)
- Tom's ACT: 24 (ACT scores: $\mu = 21$, $\sigma = 6$)

**Solution:** To compare performances on different scales, we calculate standardized z-scores:

$$z_{\text{SAT}} = \frac{X - \mu}{\sigma} = \frac{1300 - 1100}{200} = \frac{200}{200} = \boxed{1.00}$$

$$z_{\text{ACT}} = \frac{X - \mu}{\sigma} = \frac{24 - 21}{6} = \frac{3}{6} = \boxed{0.50}$$

**Ann performed better relatively.** Her SAT score is exactly one full standard deviation above the mean, while Tom's ACT score is only half a standard deviation above the mean. This means Ann's performance was more exceptional compared to the typical test-taker population.

# 3. Empirical Rule

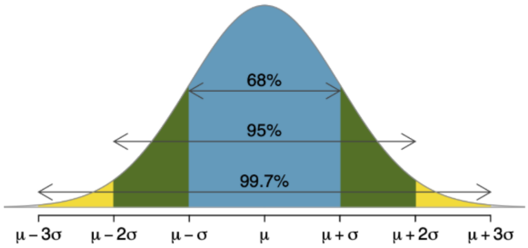


Figure 1: the 68-95-99.7 Empirical Rule

1. **Distribution:** The distribution is the **Normal distribution** (bell-shaped).
2. **Rule of thumb:** The **Empirical Rule** (also called the 68-95-99.7 rule).
3. **For each percentage:**

- **68%:** Approximately 68% of observations fall within one standard deviation of the mean ($\mu \pm 1\sigma$). This means 32% of observations fall outside this band.
- **95%:** Approximately 95% of observations fall within two standard deviations of the mean ($\mu \pm 2\sigma$). This means 5% of observations fall outside this band.
- **99.7%:** Approximately 99.7% of observations fall within three standard deviations of the mean ($\mu \pm 3\sigma$). This means 0.3% of observations fall outside this band.

# 4. Is It a Bernoulli Trial?

**Bernoulli Trial Requirements:**

1. Exactly two possible outcomes (success/failure)
2. Fixed probability of success
3. Independence between trials

**(a) Drawing poker cards without replacement:**
**Answer: No**, this is not a Bernoulli trial for two reasons:

1. **Lack of independence:** Drawing cards without replacement means each draw affects subsequent draws. For example, if the first card drawn is the ace of clubs, the probability of drawing the ace of clubs on the second draw becomes zero.
2. **Multiple outcomes:** There are 52 different possible cards (or 13 ranks, or 4 suits), not just two outcomes. To make this Bernoulli, we would need to simplify to exactly two categories (e.g., "ace" vs "not ace").

**(b) Rolling a standard six-sided die:**
**Answer: No**, because there are six possible outcomes (1, 2, 3, 4, 5, 6), not two.
**Note:** This *could* become a Bernoulli trial if we simplify to two outcomes, such as "rolling a 6" (success) vs "not rolling a 6" (failure), with $p = 1/6$.

## 5. Coffee Shop Customer Arrivals

**Answer:** Customer arrivals are best modeled by a **Poisson distribution**. The Poisson distribution is appropriate when:

- Events occur independently
- Events occur at a constant average rate
- We're counting occurrences in a fixed time interval
- The probability of multiple events in a very small time interval is negligible

Customer arrivals at a coffee shop satisfy all these conditions, with an average rate of 75 customers per hour ($\lambda = 75$).

## 6. PMF True/False

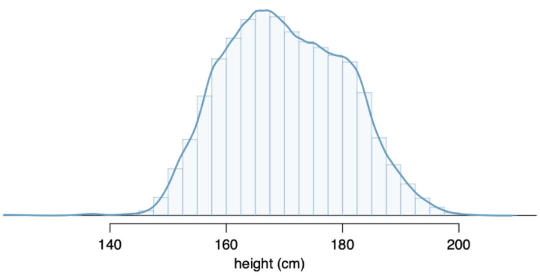


Figure 2: Probability Density Function (PDF) of Heights

**Statement:** "The curve shown represents a probability mass function (PMF) for heights."
**Answer:** **False** Heights are measured on a **continuous scale**, not a discrete scale. Therefore, the curve represents a **probability density function (PDF)**, not a probability mass function (PMF).
**Key distinction:**

- **PMF:** Used for discrete random variables (countable outcomes)
- **PDF:** Used for continuous random variables (uncountable outcomes in an interval)

## 7. Probability of Exact Height

**Question:** What is the probability that a randomly selected person is exactly 180 cm?
**Answer:** The probability is **0**. For any continuous random variable, $P(X = x) = 0$ for any specific value $x$. This is because:

- Probability corresponds to area under the PDF curve
- A single point has zero width, hence zero area
- Only intervals have positive probability: $P(a < X < b) > 0$

This is why we ask questions like "What's the probability that height is between 67.5 and 68.5 inches?" rather than asking about exact values.

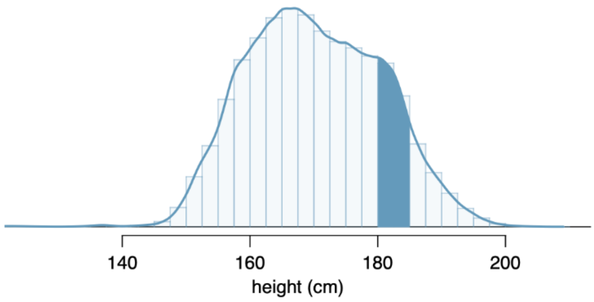


Figure 3: Distribution of height (in cm) with a shaded region representing individuals taller than 180 cm. The histogram and density curve show that height is approximately normally distributed.

# 8. Smallpox Probability Tree


Figure 4: Tree Diagram for Smallpox Data

**Calculation:** $0.0392 \times 0.9754 = 0.03824$
**Explanation:** This calculates the joint probability $P(\text{Inoculated AND Lived})$ using the multiplication rule: $P(A \cap B) = P(A) \times P(B|A)$
Where:

- $P(\text{Inoculated}) = 0.0392$
- $P(\text{Lived}|\text{Inoculated}) = 0.9754$

**Result:** The probability is $0.03824 \approx 3.82\%$. This represents the probability that a randomly selected person was both inoculated and survived.
**Note:** Tree diagrams allow us to systematically apply the multiplication rule by following branches from left to right.

# 9. Random Variable Mini-Essay

## (a) Definition

A **random variable** is a function that assigns a numerical value to each possible outcome of a random experiment or chance process. It transforms the sample space of an experiment into numerical values that we can analyze using probability theory and statistics.

## (b) Two Types

**Discrete Random Variables:**

- Take on countable values (finite or countably infinite)
- Have gaps between possible values
- Described by probability mass functions (PMF)
- Example values: $\{0, 1, 2, 3, ...\}$ or $\{1, 4, 9, 16, ...\}$

**Continuous Random Variables:**

- Take on uncountably infinite values in an interval
- Fill intervals with no gaps
- Described by probability density functions (PDF)
- Example: any real number in $(0, 1)$ or $(-\infty, \infty)$

## (c) Examples

**Discrete Example:** Let $X$ = number of red lights encountered during a 5-block commute.

- Possible values: $X \in \{0, 1, 2, 3, 4, 5\}$
- Countable outcomes with clear gaps between values

**Continuous Example:** Let $Y$ = water depth (in meters) at a randomly chosen point in Lake Tahoe.

- Possible values: $Y \in (0, 501)$ (assuming maximum depth is 501m)
- Any real number in this interval is theoretically possible

## (d) Suitable Probability Distributions

**For $X$ (red lights): Binomial distribution** with parameters $n = 5$ and $p$.

- Each intersection represents an independent Bernoulli trial
- Fixed number of trials ($n = 5$ intersections)
- Constant probability $p$ of encountering a red light
- We're counting successes (red lights) out of $n$ trials

**For $Y$ (lake depth): Normal distribution** could be appropriate.

- Lake depth results from many geological factors
- Central Limit Theorem suggests that the sum of many small, independent effects tends toward normality
- Provides a reasonable model for natural phenomena with variation around a central tendency